Symplectic twistor operator and its solution space on $\mathbb{R}^2$

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Abstract

We introduce the symplectic twistor operator T_s in symplectic spin geometry, as a symplectic analogue of the twistor operator in Riemannian spin geometry. We focus on the real dimension 2 and compute the space of its solutions on $\mathbb{R}^2$. Our analysis is based on the techniques of metaplectic Howe duality.

Key words: Symplectic spin geometry, Metaplectic Howe duality, Symplectic twistor operator, Symplectic Dirac operator.

MSC classification: 53C27, 53D05, 81R25.

1 Introduction and Motivation

Central problems and questions in differential geometry of Riemannian spin manifolds are usually reflected in analytic and spectral properties of two first order differential operators acting on spinors, the Dirac operator and the twistor operator. In particular, there is a quite subtle relation between geometry and topology of a given manifold and the spectra resp. the solution spaces of these operators, see e.g. [5], [1] and references therein.

Using the Segal-Shale-Weil representation, the symplectic version of Dirac operator D_s was introduced in [9], and some of its basic analytic and spectral properties were studied in [4], [7], [8]. Introducing the metaplectic Howe duality, [2], a representation theoretical characterization of the solution space of symplectic Dirac operator was determined on the symplectic space $(\mathbb{R}^{2n}, \omega)$. However, an explicit analytic description of this space is still missing and this fact has also substantial consequences for the present article.

The aim of the present article is to introduce symplectic twistor operator T_s, and using the metaplectic Howe duality to determine its solution space on the simplest non-compact symplectic space $(\mathbb{R}^2, \omega)$. In particular, our approach exploits a neat interplay between symplectically invariant twistor operator T_s and the generators of the Howe dual pair D_s, X_s, and a detailed analysis of each $mp(2,\mathbb{R})$-irreducible subspace of the whole function space of polynomial symplectic spinors. A part of the solution

of the problem of finding solution space of T_s is the discovery of certain canonical representative solutions of the symplectic Dirac operator D_s.

The system of partial differential equations representing T_s is overdetermined, acting on the space of functions valued in an infinite dimensional vector space of the Segal-Shale-Weil representation, and the solution space of T_s is (even locally) infinite dimensional. The geometrical meaning of the solutions of the symplectic twistor operator is not completely clear, but our analysis shows a close relationship between solutions of the symplectic twistor operator and spectral properties of the symplectic Dirac operator. In the specific case of $\mathbb{R}^2 \simeq \mathbb{C}$, the symplectic Dirac resp. twistor operators give a symplectic variant of the Cauchy-Riemann (or, Dolbeault) operator. Notice that the techniques of the metaplectic Howe duality are not restricted to $(\mathbb{R}^2, \omega)$, but it is not straightforward for $(\mathbb{R}^{2n}, \omega)$, $n > 1$ to write more explicit formulas for solutions with values in the higher dimensional non-commutative Weyl algebra.

The structure of our article goes as follows. In the first Section we review basic properties of symplectic spin geometry in dimension 2, with emphasis on metaplectic Howe duality. In the second Section we give a general definition of symplectic twistor operator T_s. The space of polynomial solutions of T_s on $\mathbb{R}^2$ is analyzed in Section three, relying on two basic principles. The first one is representation theoretical, coming from the action of the metaplectic Lie algebra on function space of interest. The second one is then the construction of representative solutions in particular irreducible subspaces of the function space. As a byproduct of our approach, we construct specific polynomial solutions of the symplectic Dirac operator D_s, which is also a novelty. In the last Section 4 we indicate the collection of unsolved problems directly related to the topic of the present article.

Throughout the article, we use the notation $\mathbb{N}_0$ for the set of natural numbers including zero and $\mathbb{N}$ for the set of natural numbers without zero.

1.1 Metaplectic Lie algebra $mp(2,\mathbb{R})$, symplectic Clifford algebra and a class of simple lowest weight modules for $mp(2,\mathbb{R})$

In the present section we recollect basic algebraic and representation theoretical information needed in the analysis of the solution space of the symplectic twistor operator T_s, see e.g., [2], [4], [6], [7], [8].

Let us consider a 2-dimensional symplectic vector space $(\mathbb{R}^2, \omega = dx \wedge dy)$, and a symplectic basis $\{e, f\}$ with respect to the non-degenerate two form $\omega \in \wedge^2 \mathbb{R}^{2\star}$. The linear action of $sp(2,\mathbb{R}) \simeq sl(2,\mathbb{R})$ on $\mathbb{R}^2$ induces action on its tensor representations, and we have $g^\star \omega = \omega$ for all $g \in sp(2,\mathbb{R})$. The set of three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is a basis of $sp(2,\mathbb{R})$.

The metaplectic Lie algebra $mp(2,\mathbb{R})$ is the Lie algebra of the twofold group covering $\pi : Mp(2,\mathbb{R}) \to Sp(2,\mathbb{R})$ of the symplectic Lie group

$Sp(2,\mathbb{R})$. It can be realized by homogeneity two elements in the symplectic Clifford algebra $Cl_s(\mathbb{R}^2,\omega)$, where the homomorphism $\pi_\star : mp(2,\mathbb{R}) \to sp(2,\mathbb{R})$ is given by

$$\begin{aligned} &\pi_\star(e\cdot e) = -2X, \\ &\pi_\star(f\cdot f) = 2Y, \\ &\pi_\star(e\cdot f + f\cdot e) = 2H. \end{aligned} \tag{1}$$

Definition 1.1 *The symplectic Clifford algebra $Cl_s(\mathbb{R}^2,\omega)$ is an associative unital algebra over $\mathbb{C}$, realized as a quotient of the tensor algebra $T(e,f)$ by a two-sided ideal $< I > \subset T(e,f)$, generated by*

$$v_i\cdot v_j - v_j\cdot v_i = -2\omega(v_i,v_j)$$

for all $v_i, v_j \in \mathbb{R}^2$.

The symplectic Clifford algebra $Cl_s(\mathbb{R}^2,\omega)$ is isomorphic to the Weyl algebra W_2 of complex valued algebraic differential operators on the real line $\mathbb{R}$, and the symplectic Lie algebra $sp(2,\mathbb{R})$ can be realized as a subalgebra of W_2. In particular, the Weyl algebra is an associative algebra generated by $\{q,\partial_q\}$, the multiplication operator by q and differentiation ∂_q, and the symplectic Lie algebra $sp(2,\mathbb{R})$ has a basis $\{-\frac{i}{2}q^2, -\frac{i}{2}\frac{\partial^2}{\partial q^2}, q\frac{\partial}{\partial q}+\frac{1}{2}\}$.

The symplectic spinor representation is an irreducible Segal-Shale-Weil representation of $Cl_s(\mathbb{R}^2,\omega)$ on $L^2(\mathbb{R}, e^{-\frac{q^2}{2}}dq_\mathbb{R})$, the space of square integrable functions on $(\mathbb{R}, d\mu = e^{-\frac{q^2}{2}}dq_\mathbb{R})$ with $dq_\mathbb{R}$ the Lebesgue measure. Its action, the symplectic Clifford multiplication c_s, preserves the subspace of C^∞(smooth)-vectors given by Schwartz space $S(\mathbb{R})$ of rapidly decreasing complex valued functions on $\mathbb{R}$ as a dense subspace. The space $S(\mathbb{R})$ can be regarded as a smooth (Frechet) globalization of the space of $\tilde{K} = \widetilde{U}(1)$-finite vectors in the representation, where $\tilde{K} \subset Mp(2,\mathbb{R})$ is the maximal compact subgroup given by the double cover of $K = U(1) \subset Sp(2,\mathbb{R})$. Though we shall work in the smooth globalization $S(\mathbb{R})$, our representative vectors constructed in Section 3 will always belong to the underlying Harish-Chandra module of $\tilde{K} = \widetilde{U}(1)$-finite vectors preserved by c_s, too. The function spaces associated to Segal-Shale-Weil representation are supported on $\mathbb{R} \subset \mathbb{R}^2$, a maximal isotropic subspace of $(\mathbb{R}^2,\omega)$.

In its restriction to $mp(2,\mathbb{R})$, it decomposes into two unitary representations realized on the subspace of even resp. odd functions:

$$\varrho : mp(2,\mathbb{R}) \to End(S(\mathbb{R})), \tag{2}$$

where the basis vectors act by

$$\begin{aligned} &\varrho(e\cdot e) = iq^2, \\ &\varrho(f\cdot f) = -i\partial_q^2, \\ &\varrho(e\cdot f + f\cdot e) = q\partial_q + \partial_q q. \end{aligned} \tag{3}$$

In this representation $Cl_s(\mathbb{R}^2,\omega)$ acts on $L^2(\mathbb{R}, e^{-\frac{q^2}{2}}dq_\mathbb{R})$ by continuous unbounded operators with domain $S(\mathbb{R})$. The space of $\tilde{K} = \widetilde{U}(1)$-finite

vectors has a basis $\{q^j e^{-\frac{q^2}{2}}\}_{j=0}^{\infty}$, its even $mp(2,\mathbb{R})$-submodule $\{q^{2j} e^{-\frac{q^2}{2}}\}_{j=0}^{\infty}$ resp. odd $mp(2,\mathbb{R})$-submodule $\{q^{2j+1} e^{-\frac{q^2}{2}}\}_{j=0}^{\infty}$. It is also an irreducible representation of $mp(2,\mathbb{R}) \ltimes h(2)$, the semidirect product of $mp(2,\mathbb{R})$ and a 3-dimensional Heisenberg Lie algebra spanned by $\{e, f, Id\}$. In the article we denote the Segal-Shale-Weil representation by $\mathcal{S}$ and we have $\mathcal{S} \simeq \mathcal{S}_+ \oplus \mathcal{S}_-$ as $mp(2,\mathbb{R})$-module.

Let us denote by $Pol(\mathbb{R}^2)$ the vector space of complex valued polynomials on $\mathbb{R}^2$, and by $Pol_l(\mathbb{R}^2)$ the subspace of homogeneity l polynomials. The complex vector space $Pol_l(\mathbb{R}^2)$ is as an irreducible $mp(2,\mathbb{R})$-module isomorphic to $S^l(\mathbb{C}^2)$, the l-th symmetric power of the complexification of the fundamental vector representation $\mathbb{R}^2$, $l \in \mathbb{N}_0$.

1.2 Segal-Shale-Weil representation and the metaplectic Howe duality

Let us recall a representation-theoretical result of [3], formulated in the opposite convention of highest weight metaplectic modules. Let λ_1 be the fundamental weight of the Lie algebra $sp(2,\mathbb{R})$, and let $L(\lambda)$ denotes the simple module over universal enveloping algebra $\mathcal{U}(mp(2,\mathbb{R}))$ of $mp(2,\mathbb{R})$ generated by highest weight vector of the weight λ. Then the Segal-Shale-Weil representation for $mp(2,\mathbb{R})$ is the highest weight representation $L(-\frac{1}{2}\lambda_1) \oplus L(-\frac{3}{2}\lambda_1)$. The highest weight vector is the eigenvector of the generator of 1-dimensional maximal commutative subalgebra of $mp(2,\mathbb{R})$.

The decomposition of the space of polynomial functions on $\mathbb{R}^2$ valued in the Segal-Shale-Weil representation corresponds to the tensor product of $L(-\frac{1}{2}\lambda_1) \oplus L(-\frac{3}{2}\lambda_1)$ with symmetric powers $S^l(\mathbb{C}^{2n})$, $l \in \mathbb{N}_0$, of the fundamental vector representation $\mathbb{C}^2$ of $sp(2,\mathbb{R})$. Note that all summands in the decomposition are again irreducible representations of $mp(2,\mathbb{R})$.

Lemma 1.2 *([3]) Let $l \in \mathbb{N}_0$.*

1. *We have for $L(-\frac{1}{2}\lambda_1)$ and all l:*

$$
\begin{aligned}
L(-\frac{1}{2}\lambda_1) \otimes S^l(\mathbb{C}^2) \quad \simeq \quad & L(-\frac{1}{2}\lambda_1) \oplus L(\lambda_1 - \frac{1}{2}\lambda_1) \oplus \dots \\
& \oplus L((2l-1)\lambda_1 - \frac{1}{2}\lambda_1) \oplus L(2l\lambda_1 - \frac{1}{2}\lambda_1),
\end{aligned}
$$

2. *We have for $L(-\frac{3}{2}\lambda_1)$ and all l:*

$$
\begin{aligned}
L(-\frac{3}{2}\lambda_1) \otimes S^l(\mathbb{C}^2) \quad \simeq \quad & L(-\frac{3}{2}\lambda_1) \oplus L(\lambda_1 - \frac{3}{2}\lambda_1) \oplus \dots \\
& \oplus L((2l-1)\lambda_1 - \frac{3}{2}\lambda_1) \oplus L(2l\lambda_1 - \frac{3}{2}\lambda_1),
\end{aligned}
$$

Another way of realizing this decomposition is the content of metaplectic Howe duality, [2]. The metaplectic analogue of the classical theorem on the separation of variables allows to decompose the space $Pol(\mathbb{R}^2) \otimes \mathcal{S}$ of complex polynomials valued in the Segal-Shale-Weil representation under

the action of $mp(2,\mathbb{R})$ into a direct sum of simple lowest weight $mp(2,\mathbb{R})$-modules

$$Pol(\mathbb{R}^2) \otimes \mathcal{S} \simeq \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{\infty} X_s^j M_l, \tag{4}$$

where we use the notation $M_l := M_l^+ \oplus M_l^-$. This decomposition takes the form of an infinite triangle

$$\begin{array}{ccccccccccc}
P_0 \otimes \mathcal{S} & & P_1 \otimes \mathcal{S} & & P_2 \otimes \mathcal{S} & & P_3 \otimes \mathcal{S} & & P_4 \otimes \mathcal{S} & & P_5 \otimes \mathcal{S} \\
\| & & \| & & \| & & \| & & \| & & \| \\
M_0 & \longrightarrow & X_s M_0 & \longrightarrow & X_s^2 M_0 & \longrightarrow & X_s^3 M_0 & \longrightarrow & X_s^4 M_0 & \longrightarrow & X_s^5 M_0 \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
 & & M_1 & \longrightarrow & X_s M_1 & \longrightarrow & X_s^2 M_1 & \longrightarrow & X_s^3 M_1 & \longrightarrow & X_s^4 M_1 \\
 & & & & \oplus & & \oplus & & \oplus & & \oplus \\
 & & & & M_2 & \longrightarrow & X_s M_2 & \longrightarrow & X_s^2 M_2 & \longrightarrow & X_s^3 M_2 \\
 & & & & & & \oplus & & \oplus & & \oplus \\
 & & & & & & M_3 & \longrightarrow & X_s M_3 & \longrightarrow & X_s^2 M_3 \\
 & & & & & & & & \oplus & & \oplus \\
 & & & & & & & & M_4 & \longrightarrow & X_s M_4 \\
 & & & & & & & & & & \oplus \\
 & & & & & & & & & & M_5
\end{array} \tag{5}$$

Let us now explain the notation used in the previous scheme. First of all, we used the shorthand notation $P_l = Pol_l(\mathbb{R}^2)$, $l \in \mathbb{N}_0$, and all spaces and arrows on the picture have the following meaning. The three operators ($i \in \mathbb{C}$ is the complex unit)

$$\begin{aligned}
X_s &= y\partial_q + ixq, \\
D_s &= iq\partial_y - \partial_x\partial_q, \\
E &= x\partial_x + y\partial_y,
\end{aligned} \tag{6}$$

where D_s acts horizontally as X_s but in the opposite direction, fulfill the $sl(2,\mathbb{R})$-commutation relations:

$$\begin{aligned}
[E+1, D_s] &= -D_s, \\
[E+1, X_s] &= X_s, \\
[D_s, X_s] &= E+1.
\end{aligned} \tag{7}$$

Let $s(x,y,q) \in Pol(\mathbb{R}^2) \otimes \mathcal{S}$, $h \in \mathrm{Mp}(2,\mathbb{R})$ and $\pi(h) = g \in \mathrm{Sp}(2,\mathbb{R})$ for the double cover map $\pi : \mathrm{Mp}(2,\mathbb{R}) \to \mathrm{Sp}(2,\mathbb{R})$. We define the action of $Mp(2,\mathbb{R})$ to be

$$\tilde{\varrho}(h)s(x,y,q) = \varrho(h)s(\pi(g^{-1})\begin{pmatrix} x \\ y \end{pmatrix}, q) = \varrho(h)s(dx - by, -cx + ay, q), \tag{8}$$

where ϱ acts on the Segal-Shale-Weil representation via (2). Passing to the infinitesimal action, we get the operators representing the basis elements

of $mp(2,\mathbb{R})$:

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\tilde{\varrho}(\exp(tX))s(x,y,q) &= \frac{d}{dt}\Big|_{t=0}\varrho\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}s(x-yt,y,q)\\
&= -\frac{i}{2}q^2 e^{-\frac{i}{2}tq^2}s(x-yt,y,q)\Big|_{t=0}\\
&\quad +e^{-\frac{i}{2}tq^2}\frac{d}{dt}s(x-yt,y,q)\Big|_{t=0}\\
&= \big(-\frac{i}{2}q^2 - y\frac{\partial}{\partial x}\big)s(x,y,q),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\tilde{\varrho}(\exp(tH))s(x,y,q) &= \frac{d}{dt}\Big|_{t=0}\varrho\begin{pmatrix}e^t & t\\ 0 & e^{-1}\end{pmatrix}s(xe^{-t},ye^t,q)\\
&= \frac{1}{2}e^{\frac{1}{2}t}s(xe^{-t},ye^t,qe^t)+e^{\frac{1}{2}t}\frac{d}{dt}s(xe^{-t},ye^t,qe^t)\Big|_{t=0}\\
&= \big(\frac{1}{2}-x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+q\frac{\partial}{\partial q}\big)s(x,y,q),
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\varrho}(X) = -y\frac{\partial}{\partial x}-\frac{i}{2}q^2,\ \tilde{\varrho}(Y) = -x\frac{\partial}{\partial y}-\frac{i}{2}\frac{\partial^2}{\partial q^2},\\
&\tilde{\varrho}(H) = -x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+q\frac{\partial}{\partial q}+\frac{1}{2},
\end{aligned}
\tag{9}
$$

and they satisfy commutation rules of the Lie algebra $mp(2,\mathbb{R})$:

$$
\begin{aligned}
&[\tilde{\varrho}(X),\tilde{\varrho}(Y)] = \tilde{\varrho}(H),\\
&[\tilde{\varrho}(H),\tilde{\varrho}(X)] = 2\tilde{\varrho}(X),\\
&[\tilde{\varrho}(H),\tilde{\varrho}(Y)] = -2\tilde{\varrho}(Y).
\end{aligned}
$$

Notice that we have not derived the explicit formula for $\tilde{\varrho}(Y)$, because it easily follows from the previous Lie algebra structure. Observe that the three operators preserve homogeneity in x, y. The Casimir operator $Cas \in \mathcal{U}(mp(2,\mathbb{R}))\otimes Cl_s(\mathbb{R}^2,\omega)$:

$$
Cas = \tilde{\varrho}(H)^2 + 1 + 2\tilde{\varrho}(X)\tilde{\varrho}(Y) + 2\tilde{\varrho}(Y)\tilde{\varrho}(X),
$$

acts by differential operator

$$
\begin{aligned}
Cas &= x^2\partial_x^2 + y^2\partial_y^2 + 2x\partial_x + 4y\partial_y + 2xy\partial_x\partial_y + \frac{1}{4}\\
&\quad -2xq\partial_x\partial_q + 2yq\partial_y\partial_q + 2iy\partial_x\partial_q^2 + 2ixq^2\partial_y\\
&= E_x(E_x-1) + E_y(E_y-1) + 2E_x + 4E_y + 2E_xE_y + \frac{1}{4}\\
&\quad -2E_xE_q + 2E_yE_q + 2iy\partial_x\partial_q^2 + 2ixq^2\partial_y.
\end{aligned}
\tag{10}
$$

Here we introduced the notation $\partial_x := \frac{\partial}{\partial x}$, $\partial_x := \frac{\partial}{\partial x}$ and $E_x = x\partial_x$, $E_y = y\partial_y$, $E_q = q\partial_q$ for the Euler homogeneity operators.

Lemma 1.3 *The operators X_s and D_s commute with operators $\tilde{\varrho}(X), \tilde{\varrho}(Y)$ and $\tilde{\varrho}(H)$. In other words, they are $mp(2,\mathbb{R})$ intertwining differential operators on complex polynomials valued in the Segal-Shale-Weil representation.*

Proof: For example, we have

$$[D_s, \tilde{\varrho}(H)] = iq\partial_y[\partial_y, y] + iq\partial_q[q, \partial_q] + \partial_x\partial_q[\partial_x, x] - \partial_x\partial_q[\partial_q, q] = 0, \quad (11)$$

and all remaining commutators are computed analogously. □

The action of $mp(2,\mathbb{R}) \times sl(2,\mathbb{R})$ generates the multiplicity free decomposition of the representation and the pair of Lie algebras in the product is called metaplectic Howe dual pair. The operators X_s, D_s act on the previous picture horizontally and isomorphically identify the two neighbouring $mp(2,\mathbb{R})$-modules. The modules M_l, $l \in \mathbb{N}$ on the most left diagonal are termed symplectic monogenics, and are characterized as l-homogeneous solutions of the symplectic Dirac operator D_s. Thus the decomposition is given as a vector space by tensor product of the symplectic monogenics multiplied by polynomial algebra of invariants $\mathbb{C}[X_s]$. The operator X_s maps polynomial symplectic spinors valued in the odd part of $\mathcal{S}$ into symplectic spinors valued in the even part of $\mathcal{S}$. This means that M_m^- is valued in $\mathcal{S}_-$, $X_s M_m^-$ is valued in $\mathcal{S}_+$, etc.

2 The symplectic twistor operator T_s

We start with an abstract definition of the symplectic twistor operator T_s and then specialize it to the symplectic space $(\mathbb{R}^2, \omega)$.

Definition 2.1 *Let (M, ∇, ω) be a symplectic spin manifold of dimension $2n$, ∇^s the associated symplectic spin covariant derivative and $\omega \in C^\infty(M, \wedge^2 T^\star M)$ a non-degenerate 2-form such that $\nabla\omega = 0$. We denote by*

$$\{e_1, \dots, e_{2n}\} \equiv \{e_1, \dots, e_n, f_1, \dots, f_n\}$$

a local symplectic frame. The symplectic twistor operator T_s on M is the first order differential operator T_s acting on smooth symplectic spinors $\mathcal{S}$:

$$\begin{aligned} &\nabla^s : C^\infty(M, \mathcal{S}) \longrightarrow T^\star M \otimes C^\infty(M, \mathcal{S}), \\ &T_s := P_{Ker(c)} \circ \omega^{-1} \circ \nabla^s : C^\infty(M, \mathcal{S}) \longrightarrow C^\infty(M, \mathcal{T}), \end{aligned} \quad (12)$$

where $\mathcal{T}$ is the space of symplectic twistors, $T^\star M \otimes \mathcal{S} \simeq \mathcal{S} \oplus \mathcal{T}$, given by algebraic projection

$$P_{Ker(c_s)} : T^\star M \otimes C^\infty(M, \mathcal{S}) \longrightarrow C^\infty(M, \mathcal{T})$$

on the kernel of the symplectic Clifford multiplication c_s. In the local symplectic coframe $\{\epsilon^1\}_{j=1}^{2n}$ dual to the sympectic frame $\{e_j\}_{j=1}^{2n}$ with respect to ω, we have the local formula for T_s:

$$T_s = \Big(1 + \frac{1}{n}\Big) \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^s_{e_k} + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \nabla^s_{e_k}, \quad (13)$$

where $\cdot$ is the shorthand notation for the symplectic Clifford multiplication and $i \in \mathbb{C}$ is the imaginary unit. We use the convention $\omega^{kj} = 1$ for $j = k+n$ and $k = 1, \ldots, n$, $\omega^{kj} = -1$ for $k = n+1, \ldots, 2n$ and $j = k-n$, and $\omega^{kj} = 0$ otherwise.

The symplectic Dirac operator D_s is defined as the image of the symplectic Clifford multiplication c_s.

Lemma 2.2 *The symplectic twistor operator T_s is $Mp(2n, \mathbb{R})$-invariant.*

Proof:

The property of invariance is a direct consequence of equivariance of symplectic covariant derivative and invariance of algebraic projection $P_{Ker(c_s)}$, and amounts to show that

$$T_s(\tilde{\varrho}(g)s) = \tilde{\varrho}(g)(T_s s) \tag{14}$$

for any $g \in Mp(2n, \mathbb{R})$ and $s \in C^\infty(M, \mathcal{S})$. Using the local formula (13) for T_s in a local chart $(x_1, \ldots, x_{2n})$, both sides of (14) are equal

$$\begin{aligned} &\Big(1 + \frac{1}{n}\Big) \sum_{k=1}^{2n} \epsilon^k \otimes \varrho(g) \frac{\partial}{\partial x^k} \big[s\big(\pi(g)^{-1}x\big)\big] \\ &+ \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \Big[\varrho(g) \frac{\partial}{\partial x^k} \big[s\big(\pi(g)^{-1}x\big)\big]\Big] \end{aligned}$$

and the proof follows.

$\square$

In the case $M = (\mathbb{R}^{2n}, \omega)$, the symplectic twistor operator is

$$T_s = \Big(1 + \frac{1}{n}\Big) \sum_{k=1}^{2n} \epsilon^k \otimes \frac{\partial}{\partial x^k} + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \frac{\partial}{\partial x^k}. \tag{15}$$

Lemma 2.3 *In the case of the symplectic space $(\mathbb{R}^2, \omega)$ with coordinates x, y and $\omega = dx \wedge dy$, a symplectic frame $\{e, f\}$ and its dual coframe $\{\epsilon^1, \epsilon^2\}$, the symplectic twistor operator $T_s : C^\infty(\mathbb{R}^2, \mathcal{S}) \to C^\infty(\mathbb{R}^2, \mathcal{T})$ acts on a smooth symplectic spinor $s(x, y, q) \in C^\infty(\mathbb{R}^2, \mathcal{S})$ as*

$$T_s(s) = \epsilon^1 \otimes \Big(\frac{\partial s}{\partial x} - q\frac{\partial^2 s}{\partial q \partial x} + iq^2 \frac{\partial s}{\partial y}\Big) + \epsilon^2 \otimes \Big(2\frac{\partial s}{\partial y} + i\frac{\partial^3 s}{\partial q^2 \partial x} + q\frac{\partial^2 s}{\partial q \partial y}\Big). \tag{16}$$

The next Lemma simplifies the condition on a symplectic spinor to be in the kernel of T_s.

Lemma 2.4 *A smooth symplectic spinor $s(x, y, q) \in C^\infty(\mathbb{R}^2, \mathcal{S})$ is in the kernel of T_s if and only if it fulfills the equation*

$$\Big(\frac{\partial}{\partial x} - q\frac{\partial^2}{\partial q \partial x} + iq^2 \frac{\partial}{\partial y}\Big) s = 0. \tag{17}$$

Proof:

The claim is a consequence of Lemma 2.3, because the covectors ϵ^1, ϵ^2 are linearly independent.

$\square$

Notice that $\tilde{\varrho}(X), \tilde{\varrho}(Y)$ and $\tilde{\varrho}(H)$ preserve the solution space of the twistor equation (17), i.e. if the symplectic spinor s solves (17) then $\tilde{\varrho}(X)s, \tilde{\varrho}(Y)s$ and $\tilde{\varrho}(H)s$ solve (17) . This is a consequence of $mp(2,\mathbb{R})$-invariance of the twistor operator T_s on $\mathbb{R}^2$ (in fact, the same observation is true in any dimension.) By abuse of notation, we use T_s in Section 3 to denote the operator (17) and call it symplectic twistor operator - this terminology is justified by the reduction in Lemma 2.4. In the article we work with polynomial (in x, y or $z, \bar{z}$) smooth symplectic spinors $Pol(\mathbb{R}^2, \mathcal{S})$.

3 The polynomial solution space of the symplectic twistor operator T_s on $\mathbb{R}^2$

Let us consider the complex vector space of symplectic spinor valued polynomials $Pol(\mathbb{R}^2, \mathcal{S})$, $\mathcal{S} \simeq \mathcal{S}_- \oplus \mathcal{S}_+$, together with its decomposition on irreducible subspaces with respect to the natural action of $mp(2,\mathbb{R})$. It follows from $mp(2,\mathbb{R})$-invariance of the symplectic twistor operator that it is sufficient to characterize its behaviour on any non-zero vector in an irreducible $mp(2,\mathbb{R})$-submodule, and that its action preserves the subspace of homogeneous symplectic spinors. This is what we are going to accomplish in the present section. Note that the meaning of the natural number $n \in \mathbb{N}$ used in previous sections to denote the dimension of the underlying symplectic space is different from its use in the present section.

The main technical difficulty consists of finding suitable representative smooth vectors in each irreducible $mp(2,\mathbb{R})$-subspace. We shall find a general characterizing condition for a polynomial (in variables x, y) valued in the Schwartz space $S(\mathbb{R})$ (in the variable q) as a formal power series, and the representative vectors are always conveniently chosen as polynomials (weighted by exponential $e^{-\frac{q^2}{2}}$) in q. In other words, the constructed vectors are $\tilde{K} = \widetilde{U}(1)$-finite vectors in $S(\mathbb{R})$. These representative vectors are then tested on the symplectic twistor operator T_s and the final conclusion is reached.

First of all, the constant symplectic spinors belong to the solution space of T_s. We have

Lemma 3.1

$$T_s(X_s e^{-\frac{q^2}{2}}) = T_s(ie^{-\frac{q^2}{2}} q(x+iy)) = 0, \tag{18}$$

$$T_s(X_s q e^{-\frac{q^2}{2}}) = T_s(e^{-\frac{q^2}{2}}(iq^2(x+iy)+y)) = 0. \tag{19}$$

The next Lemma is preparatory for further considerations.

Lemma 3.2 *We have for any $n \in \mathbb{N}_0$, $(X_s)^n \in End(Pol(\mathbb{R}^2, \mathcal{S}))$, the identity*

$$(X_s)^n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} A^n_{jk} y^{n-j-k} (ix)^{j+k} q^k \partial_q^{n-2j-k}. \tag{20}$$

Here $\lfloor \frac{n}{2} \rfloor$ *is the floor function applied to* $\frac{n}{2}$, *and the coefficients* $A^n_{jk} \in \mathbb{C}$ *fulfill the 4-term recurrent relation*

$$A^n_{jk} = A^{(n-1)}_{jk} + A^{(n-1)}_{j(k-1)} + (k+1)A^{(n-1)}_{(j-1)(k+1)}. \tag{21}$$

We use the normalization $A^0_{00} = 1$, *and* $A^n_{jk} \neq 0$ *only for* $n \in \mathbb{N}_0$, $j = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, *and* $k = 0, \ldots, n-2j$.

Proof:

The proof is by induction on $n \in \mathbb{N}_0$. The claim is trivial for $n = 0$, and for $n = 1$ we have

$$(X_s)^1 = A^1_{00} y\partial_q + A^1_{01} ixq,$$

where $A^1_{00} = A^0_{00} = 1$ and $A^1_{01} = A^0_{00} = 1$.

We assume that the formula holds for $n-1$ and aim to prove it for n:

$$(y\partial_q + ixq)\Big(\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2j} A^{(n-1)}_{jk} y^{n-1-j-k} (ix)^{j+k} q^k \partial_q^{n-1-2j-k} \Big)$$

$$= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2j} A^{(n-1)}_{jk} \big(y^{n-j-k} (ix)^{j+k} q^k \partial_q^{n-2j-k}$$
$$+ y^{n-1-j-k} (ix)^{j+k+1} q^{k+1} \partial_q^{n-1-2j-k} + k y^{n-j-k} (ix)^{j+k} q^{k-1} \partial_q^{n-1-2j-k} \big)$$

$$= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=0}^{n-1-2j} \big(A^{(n-1)}_{jk} + A^{(n-1)}_{j(k-1)} + (k+1) A^{(n-1)}_{(j-1)(k+1)} \big) y^{n-j-k} (ix)^{j+k} q^k \partial_q^{n-2j-k}$$
$$+ \big(A^{(n-1)}_{j(n-3)} + (n-1) A^{(n-1)}_{(j-1)(n-1)} \big) y^j (ix)^{n-j} q^{n-2j}$$
$$+ A^{(n-1)}_{(\lfloor \frac{n-1}{2} \rfloor)(n-2\lfloor \frac{n-1}{2} \rfloor - 1)} \Big(n - 2 \Big\lfloor \frac{n-1}{2} \Big\rfloor - 1 \Big) y^{\lfloor \frac{n-1}{2} \rfloor + 1} (ix)^{n - \lfloor \frac{n-1}{2} \rfloor - 1}$$

Now we apply the induction argument to the first term, the identity

$$\big(A^{(n-1)}_{j(n-3)} + (n-1) A^{(n-1)}_{(j-1)(n-1)} \big) = A^{(n-1)}_{j(n-2)}$$

to the second term, and as for the third term we take j to sum up to $\lfloor \frac{n}{2} \rfloor$ because for even n we have $\big(n - 2 \lfloor \frac{n-1}{2} \rfloor - 1 \big) = 1$ while for odd n this is equal to 0. Therefore, the previous expression equals to

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} A^n_{jk} y^{n-j-k} (ix)^{j+k} q^k \partial_q^{n-2j-k},$$

which completes the required statement.

□

Remark 3.3 *Notice that for* $j = 0$, *the solution of recurrent relation in (21) corresponds to binomial coefficients. It follows from* $A^{(n-1)}_{(-1)(k+1)} = 0$,

$$A^n_{0k} = A^{(n-1)}_{0k} + A^{(n-1)}_{0(k-1)},$$

and therefore, $A^n_{0k} = \binom{n}{k}$.

Lemma 3.4 *We have* $A^n_{1(n-2)} = \frac{n(n-1)}{2} = \binom{n}{n-2}$.

Proof:

We use the relation $A^n_{1(n-2)} = A^{(n-1)}_{1(n-2)} + A^{(n-1)}_{1(n-3)} + (n-1)A^{(n-1)}_{0(n-1)}$, where $A^{n-1}_{1(n-2)} = 0$ (because it is out of the range for the index k in the equation (21).) The proof goes by induction in n: we start with $A^2_{10} = A^1_{01} = 1$, and claim $A^n_{1(n-2)} = \frac{n(n-1)}{2}$. The induction step gives $A^{(n+1)}_{1(n-1)} = A^n_{1(n-2)} + nA^n_{0n} = \frac{n^2-n}{2} + n = \frac{n^2+n}{2}$.

□

Let us remark that the composition $T_s \circ (X_s)^n$ for $n = 2, 3$, acting on $e^{-\frac{q^2}{2}}$ and $qe^{-\frac{q^2}{2}}$, is non-vanishing. This means that some irreducible $mp(2,\mathbb{R})$-components in the decomposition (5) are not in the kernel of T_s:

$$\begin{aligned}
T_s(X_s^2 e^{-\frac{q^2}{2}}) &= e^{-\frac{q^2}{2}}(q^2x + iy + iq^2y) \neq 0, \\
T_s(X_s^2 q e^{-\frac{q^2}{2}}) &= e^{-\frac{q^2}{2}}(q^3x + iq^3y) \neq 0, \\
T_s(X_s^3 e^{-\frac{q^2}{2}}) &= e^{-\frac{q^2}{2}}(3iq^3x^2 - 6q^3xy - 3iq^3y^2) \neq 0, \\
T_s(X_s^3 q e^{-\frac{q^2}{2}}) &= e^{-\frac{q^2}{2}}(3iq^4x^2 + 6q^2xy - 6q^4xy + 3iy^2 + 6iq^2y^2 \\
&\quad - 3iq^4y^2) \neq 0, \qquad (22)
\end{aligned}$$

Lemma 3.5 *Let* $n \in \mathbb{N}_0$. *Then*

$$\begin{aligned}
T_s \circ (X_s)^n = &\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2j} A^n_{jk} \Big(i(j+k)y^{n-j-k}(ix)^{j+k-1}q^k\partial_q^{n-2j-k} \\
&+ y^{n-j-k}(ix)^{j+k}q^k\partial_x\partial_q^{n-2j-k} - i(j+k)y^{n-j-k}(ix)^{j+k-1}q^{k+1}\partial_q^{n-2j-k+1} \\
&- y^{n-j-k}(ix)^{j+k}q^{k+1}\partial_x\partial_q^{n-2j-k+1} - ik(j+k)y^{n-j-k}(ix)^{j+k-1}q^k\partial_q^{n-2j-k} \\
&- ky^{n-j-k}(ix)^{j+k}q^k\partial_x\partial_q^{n-2j-k} + i(n-j-k)y^{n-j-k-1}(ix)^{j+k}q^{k+2}\partial_q^{n-2j-k} \\
&+ iy^{n-j-k}(ix)^{j+k}q^{k+2}\partial_y\partial_q^{n-2j-k}\Big) \qquad (23)
\end{aligned}$$

In particular, $T_s((X_s)^n e^{-\frac{q^2}{2}}) \neq 0$ *and* $T_s((X_s)^n q e^{-\frac{q^2}{2}}) \neq 0$ *for all* $n > 1$.

Proof:

The proof is based on the identity in Lemma 3.2. The non-triviality of the composition is detected by the coefficient by monomial $x^{n-1}q^n e^{-\frac{q^2}{2}}$ in $T_s((X_s)^n e^{-\frac{q^2}{2}})$. It follows from the identity (23) that this coefficient is ($i \in \mathbb{C}$ is the complex unit)

$$\begin{aligned}
&i^n(A^n_{0n}n - A^n_{0n}n^2 + A^n_{1(n-2)})x^{n-1}q^n e^{-\frac{q^2}{2}} \\
=&i^n\left(\binom{n}{n}(n-n^2) + \binom{n}{n-2}\right)x^{n-1}q^n e^{-\frac{q^2}{2}} \\
=&-i^n\frac{n(n-1)}{2}x^{n-1}q^n e^{-\frac{q^2}{2}}, \qquad (24)
\end{aligned}$$

which is non-zero for all $n > 1$.

As for the action on the vector $qe^{-\frac{q^2}{2}}$, the situation is analogous. The coefficient by monomial $x^{n-1}q^{n+1}e^{-\frac{q^2}{2}}$ in $T_s((X_s)^n qe^{-\frac{q^2}{2}})$ is $-i^n\frac{n(n-1)}{2}$, which is again non-zero for all $n > 1$. The proof is complete.

□

In the next part we focus for a while on symplectic spinors given by iterative action of X_s on $\mathcal{S}_+$, and complete the task of finding all subspaces of polynomial solutions of T_s (expressed in the real variables x, y).

Lemma 3.6 *The vectors $e^{-\frac{q^2}{2}}(x+iy)^m \in Pol_m(\mathbb{R}^2, \mathcal{S}_+)$, $m \in \mathbb{N}_0$, are in the kernel of D_s, but not in the kernel of the symplectic twistor operator T_s.*

Proof:

We get by direct computation,

$$
\begin{aligned}
D_s\big(e^{-\frac{q^2}{2}}(x+iy)^m\big) =& iq\partial_y e^{-\frac{q^2}{2}}(x+iy)^m - \partial_x\partial_q e^{-\frac{q^2}{2}}(x+iy)^m \\
=& e^{-\frac{q^2}{2}}(-mq(x+iy)^{m-1} + mq(x+iy)^{m-1}) = 0, \\
T_s\big(e^{-\frac{q^2}{2}}(x+iy)^m\big) =& \partial_x e^{-\frac{q^2}{2}}(x+iy)^m - q\partial_x\partial_q e^{-\frac{q^2}{2}}(x+iy)^m \\
&+ iq^2\partial_y e^{-\frac{q^2}{2}}(x+iy)^m = e^{-\frac{q^2}{2}} m(x+iy)^{m-1} \neq 0
\end{aligned}
$$

for any natural number $m > 0$. □

Lemma 3.7 *Let $m \in \mathbb{N}_0$. Then the vectors $X_s e^{-\frac{q^2}{2}}(x+iy)^m$ in $Pol_{m+1}(\mathbb{R}^2, \mathcal{S}_+)$ are in the kernel of the twistor operator T_s.*

Proof:

We have

$$
\begin{aligned}
&T_s(X_s e^{-\frac{q^2}{2}}(x+iy)^m) = T_s(iqe^{-\frac{q^2}{2}}(x+iy)^{m+1}) \\
&= i(m+1)e^{-\frac{q^2}{2}}(q - q + q^2 - q^2)(x+iy)^m = 0.
\end{aligned}
$$

□

Remark 3.8 *The non-trivial elements in $Ker(T_s)$ are*

$$qe^{-\frac{q^2}{2}}(x+iy)^k,\ k \in \mathbb{N}_0. \tag{25}$$

The next Lemma completes the information on the behaviour of T_s for remaining $mp(2,\mathbb{R})$-modules coming from the action of X_s on $\mathcal{S}_+$.

Lemma 3.9 *For all natural numbers $n > 1$ and all $m \in \mathbb{N}_0$, we have*

$$T_s((X_s)^n e^{-\frac{q^2}{2}}(x+iy)^m) \neq 0. \tag{26}$$

Proof:

We focus on the coefficient by the monomial $x^{n-1+m}q^n e^{-\frac{q^2}{2}}$ in $T_s((X_s)^n e^{-\frac{q^2}{2}})$. It follows from (23) that the contribution to this coefficient is

$$\begin{aligned}
i^n(A^n_{0n}n - &A^n_{0n}n^2 + A^n_{1(n-2)} + A^n_{0n}m - A^n_{0n}mn)x^{n-1+m}q^n e^{-\frac{q^2}{2}} \\
&= i^n\left(\binom{n}{n}(n-n^2+m-mn) + \binom{n}{n-2}\right)x^{n-1+m}q^n e^{-\frac{q^2}{2}} \\
&= -\, i^n\frac{(n+2m)(n-1)}{2}x^{n-1+m}q^n e^{-\frac{q^2}{2}},
\end{aligned} \tag{27}$$

which is non-zero for all natural numbers $n > 1$ and all $m \in \mathbb{N}_0$.

□

Let us summarize the previous lemmas in the final Theorem.

Theorem 3.10 *The solution space of the symplectic twistor operator T_s acting on $Pol(\mathbb{R}^2, \mathcal{S}_+)$ consists of the set of $mp(2,\mathbb{R})$-modules in the boxes, realized in the decomposition of $Pol(\mathbb{R}^2, \mathcal{S}_+)$ on $mp(2,\mathbb{R})$ irreducible subspaces:*

$$\begin{array}{ccccccccccccc}
\underset{e^{-\frac{q^2}{2}}}{\boxed{M_0^+}} & \to & \boxed{X_sM_0^+} & \to & X_s^2M_0^+ & \to & X_s^3M_0^+ & \to & X_s^4M_0^+ & \to & X_s^5M_0^+ & & \dots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 & & \underset{e^{-\frac{q^2}{2}}(x+iy)}{M_1^+} & \to & \boxed{X_sM_1^+} & \to & X_s^2M_1^+ & \to & X_s^3M_1^+ & \to & X_s^4M_1^+ & & \dots \\
 & & & & \oplus & & \oplus & & \oplus & & \oplus & & \\
 & & & & \underset{e^{-\frac{q^2}{2}}(x+iy)^2}{M_2^+} & \to & \boxed{X_sM_2^+} & \to & X_s^2M_2^+ & \to & X_s^3M_2^+ & & \dots \\
 & & & & & & \oplus & & \oplus & & \oplus & & \\
 & & & & & & \underset{e^{-\frac{q^2}{2}}(x+iy)^3}{M_3^+} & \to & \boxed{X_sM_3^+} & \to & X_s^2M_3^+ & & \dots \\
 & & & & & & & & \oplus & & \oplus & & \\
 & & & & & & & & \underset{e^{-\frac{q^2}{2}}(x+iy)^4}{M_4^+} & \to & \boxed{X_sM_4^+} & & \dots \\
 & & & & & & & & & & \oplus & & \\
 & & & & & & & & & & M_5^+ & & \dots
\end{array} \tag{28}$$

Notice that non-zero representative vectors in the solution space of D_s are pictured under the spaces of symplectic monogenics.

This completes the picture in the case of $\mathcal{S}_+$. As we shall see, the representative solutions of D_s in an arbitrary homogeneity are far more complicated for $\mathcal{S}_-$ than for $\mathcal{S}_+$, which were chosen to be the powers of $(x+iy)$. A rather convenient way to simplify the presentation is to pass from the real coordinates x, y to the complex coordinates $z, \overline{z}$ for the standard complex structure on $\mathbb{R}^2$, where $\partial_x = \frac{1}{2}(\partial_z + \partial_{\bar{z}})$ and $\partial_y = \frac{i}{2}(\partial_z - \partial_{\bar{z}})$.

Lemma 3.11 *The operators X_s, D_s and T_s are in the complex coordinates $z, \bar{z}$ given by*

$$\begin{aligned}
X_s &= \frac{i}{2}\big((q-\partial_q)z + (q+\partial_q)\bar{z}\big), \\
D_s &= \frac{1}{2}\big((q+\partial_q)\partial_z + (-q+\partial_q)\partial_{\bar{z}}\big), \\
T_s &= \frac{1}{2}\big((1-q\partial_q-q^2)\partial_z + (1-q\partial_q+q^2)\partial_{\bar{z}}\big).
\end{aligned} \tag{29}$$

In the rest of the article we suppress the overall constants $\frac{i}{2}, \frac{1}{2}, \frac{1}{2}$ by X_s, D_s, T_s. The reason is that both the metaplectic Howe duality and the solution space of D_s, T_s are independent of the normalization of X_s, D_s, T_s. In other words, the representative solutions differ by a non-zero multiple, a property which has no effect on the results in the article.

We start with the characterization of elements in the solution space of D_s, both for $\mathcal{S}_+$ and $\mathcal{S}_-$.

Theorem 3.12 *1. The homogeneity $m \in \mathbb{N}_0$ in $z, \bar{z}$ symplectic spinor*

$$s = e^{-\frac{q^2}{2}} q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big), \tag{30}$$

with coefficients in the formal power series in q,

$$A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots,\ a_k^r \in \mathbb{C},\ r = 0, \ldots, m,\ k \in 2\mathbb{N}_0$$

is in the kernel of D_s provided the coefficients a_k^r satisfy the system of recursion relations

$$\begin{aligned} 0 &= m(k+1)a_k^m + (k+1)a_k^{m-1} - 2a_{k-2}^{m-1}, \\ 0 &= (m-1)(k+1)a_k^{m-1} + 2(k+1)a_k^{m-2} - 4a_{k-2}^{m-2}, \\ &\ldots \\ 0 &= 2(k+1)a_k^2 + (m-1)(k+1)a_k^1 - 2(m-1)a_{k-2}^1, \\ 0 &= (k+1)a_k^1 + m(k+1)a_k^0 - 2ma_{k-2}^0, \end{aligned} \tag{31}$$

equivalent to

$$(m-p)(k+1)a_k^{m-p} + (p+1)(k+1)a_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0, \tag{32}$$

for all $p = 0, 1, \ldots, m-1$.

2. The homogeneity $m \in \mathbb{N}_0$ in $z, \bar{z}$ symplectic spinor s,

$$s = e^{-\frac{q^2}{2}} \big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big), \tag{33}$$

with coefficients in the formal power series in q,

$$A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots,\ a_k^r \in \mathbb{C},\ r = 0, \ldots, m,\ k \in 2\mathbb{N}_0$$

is in the kernel of D_s provided the coefficients a_k^r satisfy the system of recursion relations

$$\begin{aligned} 0 &= mka_k^m + ka_k^{m-1} - 2a_{k-2}^{m-1}, \\ 0 &= (m-1)ka_k^{m-1} + 2ka_k^{m-2} - 4a_{k-2}^{m-2}, \\ &\ldots \\ 0 &= 2ka_k^2 + (m-1)ka_k^1 - 2(m-1)a_{k-2}^1, \\ 0 &= ka_k^1 + mka_k^0 - 2ma_{k-2}^0, \end{aligned} \tag{34}$$

equivalent to

$$(m-p)ka_k^{m-p} + (p+1)ka_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0, \tag{35}$$

for all $p = 0, 1, \ldots, m-1$.

Proof:
Because

$$(q+\partial_q)e^{-\frac{q^2}{2}}qA^r(q) = e^{-\frac{q^2}{2}}[q^2+1-q^2+q\partial_q]A^r(q),$$
$$(-q+\partial_q)e^{-\frac{q^2}{2}}qA^r(q) = e^{-\frac{q^2}{2}}[-q^2+1-q^2+q\partial_q]A^r(q),$$

the action of D_s on the vector $e^{-\frac{q^2}{2}}qA^r(q)$ is

$$\begin{aligned}
&D_s\Big(e^{-\frac{q^2}{2}}q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big)\Big)\\
&\quad = e^{-\frac{q^2}{2}}\Big(z^{m-1}\big(m[1+q\partial_q]A^m(q) + [1+q\partial_q-2q^2]A^{m-1}(q)\big)\\
&\qquad\quad z^{m-2}\bar{z}\big((m-1)[1+q\partial_q]A^{m-1}(q) + 2[1+q\partial_q-2q^2]A^{m-2}(q)\big)\\
&\qquad\quad \vdots\\
&\qquad\quad z\bar{z}^{m-1}\big(2[1+q\partial_q]A^2(q) + (m-1)[1+q\partial_q-2q^2]A^1(q)\big)\\
&\qquad\quad \bar{z}^m\big([1+q\partial_q]A^1(q) + m[1+q\partial_q-2q^2]A^0(q)\big)\Big).
\end{aligned} \tag{36}$$

The action of $[1+q\partial_q]$ on $A^r(q)$ yields $\sum_{k\in 2\mathbb{N}}(k+1)a^r_k q^k$, and the action of $[1+q\partial_q-2q^2]$ on $A^r(q)$ gives $\sum_{k\in 2\mathbb{N}}((k+1)a^r_k - 2a^r_{k-2})q^k$, for all $r=0,\ldots,m$.

As for the second part, we have

$$\begin{aligned}
(q+\partial_q)e^{-\frac{q^2}{2}}A^r(q) =& e^{-\frac{q^2}{2}}[\partial_q]A^r(q),\\
(-q+\partial_q)e^{-\frac{q^2}{2}}A^r(q) =& e^{-\frac{q^2}{2}}[-2q+\partial_q]A^r(q),
\end{aligned}$$

and the rest of the proof is analogous to the first part. The proof is complete.

$\square$

Remark 3.13 *We observe that the choice of the constant $A^0(q)=a^0_0\neq 0$, i.e. $a^0_k\neq 0$ only for $k=0$, leads to the solution (polynomial in q) of the recursion relation for all coefficients in the symplectic spinor (30).*

$$\begin{aligned}
A^0(q) =& a^0_0,\\
A^1(q) =& \left(-1+\frac{2}{3}q^2\right)\binom{m}{1}a^0_0,\\
&\ldots\\
A^r(q) =& \left((-1)^r+\ldots+\frac{2^r}{(2r+1)!!}q^{2r}\right)\binom{m}{r}a^0_0,\\
&\ldots\\
A^m(q) =& \left((-1)^m+\ldots+\frac{2^m}{(2m+1)!!}q^{2m}\right)\binom{m}{m}a^0_0,
\end{aligned}$$

where $(2m+1)!! = (2m+1)\cdot(2m-1)\cdots 3\cdot 1$. In this way we get simple representative vectors in the kernel of D_s, valued in $\mathcal{S}_-$ for each homogeneity m. We have for $m=1,2,3$:

$$
\begin{aligned}
&e^{-\frac{q^2}{2}} q\Big(\Big(-1+\frac{2}{3}q^2\Big)z+\bar{z}\Big)a_0^0,\\
&e^{-\frac{q^2}{2}}\Big(q\Big(1-\frac{4}{3}q^2+\frac{4}{15}q^4\Big)z^2+\Big(-2+\frac{4}{3}q^2\Big)z\bar{z}+\bar{z}^2\Big)a_0^0,\\
&e^{-\frac{q^2}{2}}\Big(q\Big(-1+2q^2-\frac{12}{15}q^4+\frac{8}{105}q^6\Big)z^3+\Big(3-4q^2+\frac{4}{5}q^4\Big)z^2\bar{z}\\
&\qquad +(-3+2q^2)z\bar{z}^2+\bar{z}^3\Big)a_0^0.
\end{aligned}
\tag{37}
$$

The same formulas expressed in the real variables x,y:

$$
\begin{aligned}
&\frac{2}{3}e^{-\frac{q^2}{2}}\Big(q^3(x+iy)-3iqy\Big)a_0^0,\\
&\frac{4}{15}e^{-\frac{q^2}{2}}\Big(q^5(x+iy)^2+10q^3y(-ix+y)-15qy^2\Big)a_0^0,\\
&\frac{8}{105}e^{-\frac{q^2}{2}}\Big(q^7(x+iy)^3-21iq^5(x+iy)^2y-105q^3(x+iy)y^2+105iqy^3\Big)a_0^0.
\end{aligned}
\tag{38}
$$

Another observation is that for a chosen homogeneity m in $z,\bar{z}$, the highest exponent of q is at least $2m+1$ and our solution realizes this minimum. The representative symplectic monogenics valued in $\mathcal{S}_+$ were already given for each homogeneity in Lemma 3.6.

In the following Theorem we characterize the solution space for T_s separately in the even case (including both even powers of X_s acting on $\mathcal{S}_+$ and odd powers of X_s acting on $\mathcal{S}_-$) and the odd case (including both odd powers of X_s acting on $\mathcal{S}_+$ and even powers of X_s acting on $\mathcal{S}_-$.)

Theorem 3.14 *1. The homogeneity $m\in\mathbb{N}_0$ in $z,\bar{z}$ symplectic spinor*

$$
s=e^{-\frac{q^2}{2}}q\big(A^m(q)z^m+A^{m-1}(q)z^{m-1}\bar{z}+\ldots+A^1(q)z\bar{z}^{m-1}+A^0(q)\bar{z}^m\big)
\tag{39}
$$

with coefficients in the formal power series in q,

$$
A^r=a_0^r+a_2^rq^2+a_4^rq^4+\ldots,\ a_k^r\in\mathbb{C},\ r=0,\ldots,m,\ k\in 2\mathbb{N}_0,
$$

is in the kernel of the symplectic twistor operator T_s provided the coefficients a_k^r satisfy the recursion relations

$$
\begin{aligned}
0&=mka_k^m+ka_k^{m-1}-2a_{k-2}^{m-1},\\
0&=(m-1)ka_k^{m-1}+2ka_k^{m-2}-4a_{k-2}^{m-2},\\
&\ \ldots\\
0&=2ka_k^2+(m-1)ka_k^1-2(m-1)a_{k-2}^1,\\
0&=ka_k^1+mka_k^0-2ma_{k-2}^0,
\end{aligned}
\tag{40}
$$

equivalent to

$$
(m-p)ka_k^{m-p}+(p+1)ka_k^{m-1-p}-2(p+1)a_{k-2}^{m-1-p}=0,
\tag{41}
$$

for all $p=0,1,\ldots,m-1$.

2. The homogeneity $m \in \mathbb{N}_0$ in $z, \bar{z}$ symplectic spinor

$$s = e^{-\frac{q^2}{2}}\left(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\right) \tag{42}$$

with coefficients in the formal power series in q,

$$A^r = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots,\ a_k^r \in \mathbb{C},\ r = 0, \ldots, m,\ k \in 2\mathbb{N}_0,$$

is in the kernel of the symplectic twistor operator T_s provided the coefficients a_k^r satisfy the recursion relations

$$\begin{aligned} 0 &= m(k-1)a_k^m + (k-1)a_k^{m-1} - 2a_{k-2}^{m-1}, \\ 0 &= (m-1)(k-1)a_k^{m-1} + 2(k-1)a_k^{m-2} - 4a_{k-2}^{m-2}, \\ &\ldots \\ 0 &= 2(k-1)a_k^2 + (m-1)(k-1)a_k^1 - 2(m-1)a_{k-2}^1, \\ 0 &= (k-1)a_k^1 + m(k-1)a_k^0 - 2ma_{k-2}^0, \end{aligned} \tag{43}$$

equivalent to

$$(m-p)(k-1)a_k^{m-p} + (p+1)(k-1)a_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0, \tag{44}$$

for all $p = 0, 1, \ldots, m-1$.

Proof:

Concerning the first part, we have

$$\begin{aligned} &T_s\Big(e^{-\frac{q^2}{2}}q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big)\Big) \\ &= e^{-\frac{q^2}{2}}q^2\Big(z^{m-1}\left(m[-\partial_q]A^m(q) + [2q - \partial_q]A^{m-1}(q)\right) \\ &\qquad + z^{m-2}\bar{z}\left((m-1)[-\partial_q]A^{m-1}(q) + 2[2q-\partial_q]A^{m-2}(q)\right) \\ &\qquad \ldots \\ &\qquad + \bar{z}^m\left([-\partial_q]A^1(q) + m[2q-\partial_q]A^0(q)\right)\Big) = 0, \end{aligned}$$

where

$$\begin{aligned} [-\partial_q]A^r(q) &= -2a_2^r q - 4a_4^r q^3 - 6a_6^r q^5 - \ldots, \\ [2q-\partial_q]A^r(q) &= (2a_0^r - 2a_2^r)q + (2a_2^r - 4a_4^r)q^3 + \ldots, \end{aligned}$$

etc. Then the coefficients of $A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots$, $r = 0, \ldots, m$ satisfy the recursion relations

$$(m-p)ka_k^{m-p} + (p+1)ka_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0,\ p = 0, \ldots, m-1.$$

As for the second part, we get

$$\begin{aligned} (1 - q\partial_q - q^2)e^{-\frac{q^2}{2}}A^r(q) &= e^{-\frac{q^2}{2}}[1 - q\partial_q]A^r(q), \\ (1 - q\partial_q + q^2)e^{-\frac{q^2}{2}}A^r(q) &= e^{-\frac{q^2}{2}}[1 + 2q^2 - q\partial_q]A^r(q). \end{aligned} \tag{45}$$

The annihilation condition for the symplectic twistor operator T_s acting on (42) is equivalent to

$$\begin{aligned}
&T_s\Big(e^{-\frac{q^2}{2}}\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big)\Big)\\
&= e^{-\frac{q^2}{2}}\Big(z^{m-1}\big(m[1-q\partial_q]A^m(q) + [1+2q^2-q\partial_q]A^{m-1}(q)\big)\\
&\qquad z^{m-2}\bar{z}\big((m-1)[1-q\partial_q]A^{m-1}(q) + 2[1+2q^2-q\partial_q]A^{m-2}(q)\big)\\
&\qquad \vdots\\
&\qquad z\bar{z}^{m-1}\big(2[1-q\partial_q]A^2(q) + (m-1)[1+2q^2-q\partial_q]A^1(q)\big)\\
&\qquad \bar{z}^m\big([1-q\partial_q]A^1(q) + m[1+2q^2-q\partial_q]A^0(q)\big)\Big), \qquad (46)
\end{aligned}$$

and this completes the proof of the Theorem.

□

Remark 3.15 *The explicit solution vectors for the symplectic twistor operator T_s are, for the choice of $A^0(q) = a^0_0 \neq 0$, given in homogeneities $m = 1, 2, 3$ by*

$$\begin{aligned}
&e^{-\frac{q^2}{2}}\big(\left(-1+2q^2\right)z + \bar{z}\big)a^0_0,\\
&e^{-\frac{q^2}{2}}\left(\left(1-4q^2+\frac{4}{3}q^4\right)z^2 + \left(-2+4q^2\right)z\bar{z} + \bar{z}^2\right)a^0_0,\\
&e^{-\frac{q^2}{2}}\left(\left(-1+6q^2-4q^4+\frac{8}{15}q^6\right)z^3 + \left(3-12q^2+4q^4\right)z^2\bar{z}\right.\\
&\qquad\left. + \left(-3+6q^2\right)z\bar{z}^2 + \bar{z}^3\right)a^0_0.
\end{aligned}$$

The same solutions expressed in the variables x, y are

$$\begin{aligned}
&2e^{-\frac{q^2}{2}}\left(q^2(x+iy) - iy\right)a^0_0,\\
&\frac{4}{3}e^{-\frac{q^2}{2}}\left(q^4(x+iy)^2 + 6q^2y(-ix+y) - 3y^2\right)a^0_0,\\
&\frac{8}{15}e^{-\frac{q^2}{2}}\left(q^6(x+iy)^3 - 15iq^4(x+iy)^2y - 45q^2(x+iy)y^2 + 15iy^3\right)a^0_0.
\end{aligned} \qquad (47)$$

Theorem 3.16 *Let $s = s(z, \bar{z}, q) \in Pol(\mathbb{R}^2, \mathcal{S}_-)$ be a polynomial symplectic spinor in the solution space of the symplectic Dirac operator D_s, i.e. the symplectic spinor s satisfies the recursion relations in the first part of Theorem (3.12). Then $X_s(s)$ is in kernel of the symplectic twistor operator, $T_s\big(X_s(s)\big) = 0$.*

Proof:

Let us consider polynomial symplectic spinor of homogeneity m,

$$s = e^{-\frac{q^2}{2}}q\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\big),$$

where $A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots$, $r = 0, \ldots, m$ satisfies the recursive relations (32). We use the notational simplification $s(z, \bar{z}, q) = e^{-\frac{q^2}{2}} qW$, $W = W(z, \bar{z}, q)$. Then

$$X_s(e^{-\frac{q^2}{2}} qW) = e^{-\frac{q^2}{2}} \left([2q^2 - 1 - q\partial_q]zW + [1 + q\partial_q]\bar{z}W\right),$$

which can be rewritten as

$$X_s(e^{-\frac{q^2}{2}} qW) = e^{-\frac{q^2}{2}} \left(B^{m+1}(q)z^{m+1} + B^m(q)z^m\bar{z} + \ldots + B^0(q)\bar{z}^{m+1}\right),$$

where $B^r(q) = b_0^r + b_2^r q^2 + b_4^r q^4 + \ldots$, $r = 0, \ldots, m+1$, and the coefficients of this formal power series satisfy

$$b_k^m = 2a_{k-2}^{m-1} + (k+1)(a_k^m - a_k^{m-1}). \tag{48}$$

We show that $B^r(q)$ satisfy the recursion relations (44) for $p = 0, 1, \ldots, m$ in Theorem (3.14). It follows from (48) that

$$\begin{aligned}
&(m+1-p)(k-1)\left(2a_{k-2}^{m-p} + (k+1)(a_k^{m-p+1} - a_k^{m-p})\right) \\
&+ (p+1)(k-1)\left(2a_{k-2}^{m-p-1} + (k+1)(a_k^{m-p} - a_k^{m-p-1})\right) \\
&- 2(p+1)\left(2a_{k-4}^{m-p-1} + (k-1)(a_{k-2}^{m-p} - a_{k-2}^{m-p-1})\right) \\
&= 2\left((m-p)(k-1)a_{k-2}^{m-p} + (p+1)(k-1)a_{k-2}^{m-p-1} - 2(p+1)a_{k-4}^{m-p-1}\right) \\
&+ (k-1)\left((m-p+1)(k+1)a_k^{m-p+1} + p(k+1)a_k^{m-p} - 2pa_{k-2}^{m-p}\right) \\
&- (k-1)\left((m-p)(k-1)a_k^{m-p} + (p+1)(k-1)a_k^{m-p-1} - 2(p+1)a_{k-2}^{m-p-1}\right) \\
&+ 2(k-1)a_{k-2}^{m-p} - (k-1)(k+1)a_k^{m-p} + (k-1)(k+1)a_k^{m-p} - 2(k-1)a_{k-2}^{m-p} \\
&= 0,
\end{aligned} \tag{49}$$

where we used for the last equality the relation (32) to verify that each of the three rows in the last but one expression equals to zero. The proof is complete.

□

Theorem 3.17 *Let $s = s(z, \bar{z}, q) \in Pol_m(\mathbb{R}^2, \mathcal{S}_-)$ be a symplectic spinor polynomial in the solution space of the symplectic Dirac operator D_s. Then s is not in the kernel of the twistor operator T_s if and only if $m \in \mathbb{N}$.*

Proof:

By our assumption, the symplectic spinor s satisfies the recursion relation in Theorem 3.12. Recall the recursion relations for symplectic spinors valued in $\mathcal{S}_-$, which are in the solution space of $\operatorname{Ker} T_s$ (see 41):

$$(m-p)ka_k^{m-p} + (p+1)ka_k^{m-1-p} - 2(p+1)a_{k-2}^{m-1-p} = 0,\ p = 0, \ldots, m-1.$$

By Theorem 3.12, the coefficients a_k^r satisfy the relations (32)

$$(m-p)(k+1)a_k^{m-p} + (p+1)(k+1)a_k^{m-1-p} + 2(p+1)a_{k-2}^{m-1-p} = 0.$$

The comparison of the last two relations leads to

$$(m-p)a_k^{m-p} + (p+1)a_k^{m-1-p} = 0 \tag{50}$$

for all k, p, and these are just the coefficients by $q^{k+1}z^{m-1-p}\bar{z}^p$ in $T_s(s)$. We choose the symplectic monogenic s according to Remark 3.13. For $k = 2, p = 0$, the coefficient in $T_s(s)$ by $q^3\bar{z}^{m-1}$ is $\left(a_2^1 + ma_2^0\right)$. Our choice for s to be a solution for D_s gives $a_2^1 = \frac{2m}{3}a_0^0$ and $a_2^0 = 0$, therefore the coefficient in (50) will not be equal to zero and consequently will not be in $\operatorname{Ker} T_s$ for $m \in \mathbb{N}$. By $mp(2,\mathbb{R})$-invariance, the whole metaplectic module does not belong to the kernel of T_s, which finishes the proof.

□

Theorem 3.18 *Let $m \in \mathbb{N}_0, k \in 2\mathbb{N}_0$.*

1. *The recursion relations for the coefficients a_k^r of an even (even homogeneity in q) symplectic spinor s,*

   $$s = e^{-\frac{q^2}{2}}\left(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\right),$$

   $A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots$, $r = 0, \ldots, m$, which is in the kernel of the square of the symplectic Dirac operator D_s^2, are

   $$\begin{aligned}&(m-p)(m-p-1)(k+2)(k+1)a_{k+2}^{m-p}+\\&(m-1-p)(p+1)\big(2(k+2)(k+1)a_{k+2}^{m-1-p} - 2(2k+1)a_k^{m-1-p}\big)+\\&(p+1)(p+2)\big((k+2)(k+1)a_{k+2}^{m-2-p} - 2(2k+1)a_k^{m-2-p} + 4a_{k-2}^{m-2-p}\big)\\&= 0\end{aligned} \tag{51}$$

   for $p = 0, \ldots, m-2$.

2. *The recursion relations for the coefficients a_k^r of an odd (odd homogeneity in q) symplectic spinor s,*

   $$s = e^{-\frac{q^2}{2}}q\left(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^1(q)z\bar{z}^{m-1} + A^0(q)\bar{z}^m\right),$$

   $A^r(q) = a_0^r + a_2^r q^2 + a_4^r q^4 + \ldots$, $r = 0, \ldots, m$, which is in the kernel of the square of the symplectic Dirac operator D_s^2, are

   $$\begin{aligned}&(m-p)(m-p-1)(k+2)(k+3)a_{k+2}^{m-p}+\\&(m-1-p)(p+1)\big(2(k+2)(k+3)a_{k+2}^{m-1-p} - 2(2k+3)a_k^{m-1-p}\big)+\\&(p+1)(p+2)\big((k+2)(k+3)a_{k+2}^{m-2-p} - 2(2k+3)a_k^{m-2-p} + 4a_{k-2}^{m-2-p}\big)\\&= 0.\end{aligned} \tag{52}$$

   for $p = 0, \ldots, m-2$.

Proof:

The second power of the symplectic Dirac operator D_s is equal to

$$D_s^2 = (q^2 + 2q\partial_q + 1 + \partial_q^2)\partial_z^2 + 2(-q^2 + \partial_q^2)\partial_z\partial_{\bar{z}} + (q^2 - 2q\partial_q - 1 + \partial_q^2)\partial_{\bar{z}}^2. \tag{53}$$

In the even case, the action of D_s^2 results in

$$
\begin{aligned}
& D_s^2\Big(e^{-\frac{q^2}{2}}\big(A^m(q)z^m + A^{m-1}(q)z^{m-1}\bar{z} + \ldots + A^0(q)\bar{z}^m\big)\Big) \\
&= e^{-\frac{q^2}{2}}\Big(z^{m-2}\big(m(m-1)[\partial_q^2]A^m(q) + (m-1)[2\partial_q^2 - 4q\partial_q - 2]A^{m-1}(q) \\
&\qquad\quad + 2[\partial_q^2 - 4q\partial_q - 2 + 4q^2]A^{m-2}(q)\big) + \ldots + \\
&\qquad \bar{z}^{m-2}\big(2[\partial_q^2]A^2(q) + (m-1)[2\partial_q^2 - 4q\partial_q - 2]A^1(q) + \\
&\qquad\quad + m(m-1)[\partial_q^2 - 4q\partial_q - 2 + 4q^2]A^0(q)\big)\Big),
\end{aligned} \tag{54}
$$

where

$$
\begin{aligned}
[\partial_q^2]A^r(q) &= 2a_2^r + 12a_4^r q^2 + \ldots \\
[2\partial_q^2 - 4q\partial_q - 2]A^r(q) &= 4a_2^r - 2a_0^r + (24a_4^r - 8a_2^r - 2a_2^r)q^2 + \ldots \\
[\partial_q^2 - 4q\partial_q - 2 + 4q^2]A^r(q) &= 2a_2^r - 2a_0^r + (12a_4^r - 8a_2^r - 2a_2^r + 4a_0^r)q^2 + \ldots
\end{aligned}
$$

The odd homogeneity case is analogous. Denoting $s = e^{-\frac{q^2}{2}}qW$, where $W = A^m(q)z^m + \ldots + A^0(q)\bar{z}^m$, we get

$$
\begin{aligned}
\partial_z^2(q^2 + 2q\partial_q + 1 + \partial_q^2)e^{-\frac{q^2}{2}}qW &= \partial_z^2 e^{-\frac{q^2}{2}}[2\partial_q + q\partial_q^2]W, \\
2\partial_z\partial_{\bar{z}}(-q^2 + \partial_q^2)e^{-\frac{q^2}{2}}qW &= 2\partial_z\partial_{\bar{z}}e^{-\frac{q^2}{2}}[q\partial_q^2 - 2q^2\partial_q + 2\partial_q - 3q]W, \\
\partial_{\bar{z}}^2(q^2 - 2q\partial_q - 1 + \partial_q^2)e^{-\frac{q^2}{2}}qW &= \partial_{\bar{z}}^2 e^{-\frac{q^2}{2}}[q\partial_q^2 - 4q^2\partial_q + 2\partial_q + 4q^3 - 6q]W,
\end{aligned}
$$

and the proof follows.

The irreducible $mp(2,\mathbb{R})$-submodules in the kernel of D_s^2 were put into boxes on the scheme of the $mp(2,\mathbb{R})$-decomposition of $Pol(\mathbb{R}^2) \otimes \mathcal{S}$:

$$
\begin{array}{cccccc}
\boxed{M_0} \to & \boxed{X_sM_0} \to & X_s^2M_0 \to & X_s^3M_0 \to & X_s^4M_0 \to & X_s^5M_0 \\
 & \oplus & \oplus & \oplus & \oplus & \oplus \\
 & \boxed{M_1} \to & \boxed{X_sM_1} \to & X_s^2M_1 \to & X_s^3M_1 \to & X_s^4M_1 \\
 & & \oplus & \oplus & \oplus & \oplus \\
 & & \boxed{M_2} \to & \boxed{X_sM_2} \to & X_s^2M_2 \to & X_s^3M_2 \\
 & & & \oplus & \oplus & \oplus \\
 & & & \boxed{M_3} \to & \boxed{X_sM_3} \to & X_s^2M_3 \\
 & & & & \oplus & \oplus \\
 & & & & \boxed{M_4} \to & \boxed{X_sM_4} \\
 & & & & & \oplus \\
 & & & & & \boxed{M_5}
\end{array} \tag{55}
$$

$\square$

Theorem 3.19 *The solution space of the symplectic twistor operator T_s is a subspace of the space of solutions of the square of the symplectic Dirac operator D_s^2. In particular, the recursion relations for D_s^2 specialized to even resp. odd symplectic spinors from Theorem 3.18 are solved by (44) resp. (41).*

Proof:

Let us start with even symplectic spinors. It is straigtforward to rewrite the recursion relations in Theorem 3.18,

$$
\begin{aligned}
&(m-p)(m-p-1)(k+2)(k+1)a_{k+2}^{m-p}+\\
&(m-1-p)(p+1)\big(2(k+2)(k+1)a_{k+2}^{m-1-p}-2(2k+1)a_k^{m-1-p}\big)+\\
&(p+1)(p+2)\big((k+2)(k+1)a_{k+2}^{m-2-p}-2(2k+1)a_k^{m-2-p}+4a_{k-2}^{m-2-p}\big)=0,
\end{aligned}
$$

into

$$
\begin{aligned}
&(m-1-p)(k+2)\big((m-p)(k+1)a_{k+2}^{m-p}+(p+1)(k+1)a_{k+2}^{m-1-p}-2(p+1)a_k^{m-1-p}\big)\\
&+(p+1)(k+2)\big((m-1-p)(k+1)a_{k+2}^{m-1-p}+(p+2)(k+1)a_{k+2}^{m-2-p}-2(p+2)a_k^{m-2-p}\big)\\
&-2(p+1)\big((m-1-p)(k-1)a_k^{m-1-p}+(p+2)(k-1)a_k^{m-2-p}-2(p+2)a_{k-2}^{m-2-p}\big)=0.
\end{aligned}
$$

Because each of the last three rows corresponds to a recursion relation (44), the claim follows.

In the odd case, the recursion relations

$$
\begin{aligned}
&(m-p)(m-p-1)(k+2)(k+3)a_{k+2}^{m-p}+\\
&(m-1-p)(p+1)\big(2(k+2)(k+3)a_{k+2}^{m-1-p}-2(2k+3)a_k^{m-1-p}\big)+\\
&(p+1)(p+2)\big((k+2)(k+3)a_{k+2}^{m-2-p}-2(2k+3)a_k^{m-2-p}+4a_{k-2}^{m-2-p}\big)=0,
\end{aligned}
$$

can be rewritten as

$$
\begin{aligned}
&(m-1-p)(k+3)\big((m-p)(k+2)a_{k+2}^{m-p}+(p+1)(k+2)a_{k+2}^{m-1-p}-2(p+1)a_k^{m-1-p}\big)\\
&+(p+1)(k+3)\big((m-1-p)(k+2)a_{k+2}^{m-1-p}+(p+2)(k+2)a_{k+2}^{m-2-p}-2(p+2)a_k^{m-2-p}\big)\\
&-2(p+1)\big((m-1-p)ka_k^{m-1-p}+(p+2)ka_k^{m-2-p}-2(p+2)a_{k-2}^{m-2-p}\big)=0,
\end{aligned}
$$

and each of the last three rows corresponds to the recursion relation (41).

□

Theorem 3.20 *The solution space of the symplectic twistor operator T_s, acting on $Pol(\mathbb{R}^2,\mathcal{S})$, consists of the set of $mp(2,\mathbb{R})$-modules pictured in the squares realized in the decomposition of $Pol(\mathbb{R}^2,\mathcal{S})$ on $mp(2,\mathbb{R})$ irreducible subspaces, (5):*

1. *$Pol(\mathbb{R}^2,\mathcal{S}_-)$:*

$$
\begin{array}{ccccccccc}
\boxed{M_0^-} & \longrightarrow & \boxed{X_sM_0^-} & \longrightarrow & X_s^2M_0^- & \longrightarrow & X_s^3M_0^- & \longrightarrow & \dots \\
qe^{-\frac{q^2}{2}} & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^- & \longrightarrow & \boxed{X_sM_1^-} & \longrightarrow & X_s^2M_1^- & \longrightarrow & \dots \\
 & e^{-\frac{q^2}{2}}(q^3(x+iy)-3iqy) & & & \oplus & & \oplus & & \\
 & & & & M_2^- & \longrightarrow & \boxed{X_sM_2^-} & \longrightarrow & \dots \\
 & e^{-\frac{q^2}{2}}(q^5(x+iy)^2+10q^3y(-ix+y)-15qy^2) & & & & & \oplus & & \\
 & & & & & & M_3^- & \longrightarrow & \dots
\end{array}
\tag{56}
$$

2. $Pol(\mathbb{R}^2, \mathcal{S}_+)$:

$$\begin{tikzcd}[row sep=tiny]
\boxed{M_0^+} \arrow[r] & \boxed{X_s M_0^+} \arrow[r] & X_s^2 M_0^+ \arrow[r] & X_s^3 M_0^+ \arrow[r] & \dots \\
e^{-\frac{q^2}{2}} & \oplus & \oplus & \oplus & \\
& M_1^+ \arrow[r] & \boxed{X_s M_1^+} \arrow[r] & X_s^2 M_1^+ \arrow[r] & \dots \\
& e^{-\frac{q^2}{2}}(x+iy) & \oplus & \oplus & \\
& & M_2^+ \arrow[r] & \boxed{X_s M_2^+} \arrow[r] & \dots \\
& & e^{-\frac{q^2}{2}}(x+iy)^2 & \oplus & \\
& & & M_3^+ \arrow[r] & \dots
\end{tikzcd} \tag{57}$$

Notice that the representative vectors in the solution space of D_s are pictured under the spaces of symplectic monogenics. In the case of $\mathcal{S}_+$, we exploit the symplectic monogenics constructed in Theorem 3.10.

Proof: It follows from the metaplectic Howe duality, [2], that Theorem 3.19 characterizes the $mp(2, \mathbb{R})$-submodule of $Pol(\mathbb{R}^2, \mathcal{S})$ containing solution space of T_s. Then Theorem 3.10, Theorem 3.17 and Theorem 3.18 characterize the space of solutions as the image of the space of symplectic monogenics by X_s, in addition to the space of constant symplectic spinors. The proof is complete.

$\square$

In previous sections, we discussed the space of polynomial solutions. A natural question is an extension of the function space from polynomials to the class of analytic, smooth, hyperfunction, generalized, etc., function spaces. For example, one can consider convergent power series constructed from the polynomial solutions. We shall not attemt to discuss this question in a wider generality, but observe the existence of a wider class of solutions.

Let us consider the function element $z^n f(q)$ for $f \in S(\mathbb{R})$, $n \in \mathbb{N}_0$. The substitution into (17) implies that it belongs to the solution space of T_s provided $f(q)$ solves the ordinary differential equation

$$(1 - q^2)f(q) = q\frac{\partial}{\partial q}f(q). \tag{58}$$

This equation has a unique solution $f(q) = qe^{-\frac{q^2}{2}}$ in $S(\mathbb{R})$, and so $z^n qe^{-\frac{q^2}{2}}$ are in the kernel of the symplectic twistor operator for all $n \in \mathbb{N}_0$.

A generalization of this result is contained in the following lemma.

Lemma 3.21 *Let $h(z)$ be an arbitrary holomorphic function on $\mathbb{C}$. Then the complex analytic symplectic spinor*

$$h(z)qe^{-\frac{q^2}{2}} \tag{59}$$

is in the kernel of the symplectic twistor operator T_s.

Consequently, the space of holomorphic functions on $\mathbb{C}$ is embedded into the space of smooth solutions of the symplectic twistor operator T_s.

4 Open problems and questions

Here we comment on several issues related to the symplectic twistor operator T_s, unresolved in the present article.

A complete understanding of the solution space of both T_s and D_s is related to writing explicit solution of the recursion relation (21). Notice that a well-known identity in the Weyl algebra,

$$(q\partial_q)^n = \sum_{m=1}^{n} s(n,m) q^m \partial_q^m, \tag{60}$$

involves Stirling number fulfilling the 3-term recursion relation

$$s(n,m) = ms(n-1,m) + s(n-1,m-1),\ m,n \in \mathbb{N}. \tag{61}$$

Our problem involves another identity in the Weyl algebra. Namely, let us introduce the variables $q, \partial_q, \tilde{q}$ fulfilling

$$[\partial_q, q] = \tilde{q},\ [\partial_q, \tilde{q}] = 0,\ [q, \tilde{q}] = 0,$$

and define

$$(q + \partial_q)^n = \sum_{r=0}^{min(i,n-i)} \sum_{i=0}^{n} \tilde{s}(n,i,r) q^{i-r} \partial_q^{n-i-r} \tilde{q}^r$$

fulfilling the 4-term recursion relation (21). As an example, we have

$$\begin{aligned} n = 2 &: \ q^2 + \partial_q^2 + 2q\partial_q + \tilde{q}, \\ n = 3 &: \ q^3 + \partial_q^3 + 3q\partial_q^2 + 3q^2\partial_q + 3\tilde{q}\partial_q + 3\tilde{q}q. \end{aligned} \tag{62}$$

It seems that $\tilde{s}(n,i,r)$, its generating functions or their closed formulas for all n, i, r were not studied in combinatorial number theory.

Another question is related to the representation theoretical problem of globalization of a given representation. Notice that an admissible continuous representation spaces of a reductive Lie group G can be conveniently described in terms of a globalization of the underlying Harish-Chandra $(\mathfrak{g}, K)$-module, where $\mathfrak{g}$ resp. K are the Lie algebra resp. maximal compact subgroup of G. In this way, one has continuous representation of G on the space of analytic, smooth, Frechet, hyperfunction, generalized, etc., functions. However, in our case of $G = Mp(2,\mathbb{R})$, $\mathfrak{g} = mp(2,\mathbb{R})$ and K given by the twofold covering of $U(1)$, the representation on symplectic spinors is not admissible - in its composition series there are infinite multiplicities of certain G-representations. This means that the functional analytic tools developed in representation theory are not straightforward to apply in our case. On the other hand, it is still natural to ask for a characterization of the space of analytic, smooth, Frechet, hyperfunction, generalized, etc., solutions of both T_s and D_s.

The present approach is applicable in many other cases and situations of interest, e.g. to higher symplectic twistor operators or higher spin analogues of the symplectic Dirac operator. Another observation is that the metaplectic Howe duality holds true in any even dimension, and so at

least conjecturally, many results carry over from dimension 2 to any even dimension $2n$, $n \in \mathbb{N}$.

Another issue is the close relationship between Riemannian and conformal structures, especially the existence of the conformal Lie group acting on function spaces as an organizing principle for subspaces acted upon by the Lie group of rotations. In particular, it is not clear to the authors whether there is a relation with the subject of conformally symplectic structure, [10], or to consider an action of the overgroup $Mp(2n+2,\mathbb{R}) \supset Mp(2n,\mathbb{R})$ in the framework of the prolongation of overdetermined system of PDEs in parabolic invariant theory, [8].

A natural question of the influence of local geometry and global topology (involving the compact examples of Riemann surfaces like the sphere, torus, etc.) on the space of solutions of T_s is under current investigation.

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